SimCalc Affordances for Democratizing Access to Advanced Mathematics

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Abstract

Historically, what people can learn is co-determined by the representational infrastructure for knowledge building. When Latin was the required medium of knowledge building, few could engage in scholarly activities; without the change to the vernacular, nearly universal access to higher education would not be possible. The highly compact, abstract, and opaque symbolism of mathematics presents similar barriers to the necessary democratization of access to important mathematics.

Over the course of a program of research lasting more than 20 years and involving contributors from institutions throughout the United States and worldwide [25], the representationally innovative design of SimCalc Mathworlds® has provided affordances for novel and effective approaches to teaching important algebraic and calculus-related ideas. When integrated with appropriate curricular workbooks, teacher professional development, and other instructional factors, dynamic representation has enabled diverse populations to learn more advanced mathematics. Research has included both design research as well as large-scale experiments involving hundreds of teachers and thousands of students; overall, the approach also has an unusually strong base of empirical support.

We focus on lasting, essential design contributions of this body of work with a special emphasis on the dialectic relationship between affordances of technology and curricular progressions. For example, the ability of technology to easily support manipulation of piecewise linear graphs enables the introduction of key mathematics much earlier in the curriculum. Likewise, the ability of technology to link evocative “familiar” experiences (running across a soccer field) with mathematical representations (a dynamic graph) enables a reversal in the normal “learn math then apply it” sequence; students can be empowered to learn in a progression from the familiar to the formal. Further, the provision to students of a mathematical medium which they find expressive for telling stories and other student-centered purposes can make doing mathematics more personally meaning-
ful. All of this has been possible by advances in the representational infrastructure of computational devices in concert with the communication infrastructure widely available in classrooms. This has allowed students to pass and share their constructions between themselves and the teacher enhancing the mathematical discourse in personally meaningful ways.

I. Intellectual Background

IA. LOGO and the Concept of Microworlds

One of the great promises of computer technology in education has been transformation. Some projects try to change learning by, for example, liberating learning from the classroom; others by liberating learning from teachers; yet others, as in the SimCalc project\(^1\), have sought to use technology to liberate mathematics learning from arcane, esoteric symbol systems and render it more readily approachable and understandable.

Historical roots of the Simcalc Mathworlds® approach built upon an early great educational movement that was based on the Logo computer language for children. Logo, promoted and memorialized in Seyour Papert’s 1980 book, Mindstorms [46], contrasts with the earlier “CAI” (Computer-Assisted Instruction), approach as represented by PLATO [18]. Logo was built to be learnable along principles influenced by the great Swiss psychologist Jean Piaget; Papert offered the evocative analogy from how he explored the concept of ratio as a child using physical gears to how children could now explore a broader range of mathematical concepts using Logo as “gears for the mind.” PLATO, in contrast, automated a traditional instructional approach consisting of providing the student with information, practice tasks, and feedback. Whereas PLATO offered teachers an approach to authoring instruction, Logo sought to offer children opportunities to construct their own computer programs. But Logo was more than a computer language.

The expressive form of turtle geometry allowed children to explore a rich panoply of outcomes related to the details of their programs. Programs could control the actions of a physical or virtual turtle, by asking it to, for example, move forward. A physical or virtual pen left a trail, thus allowing the children to at once draw and picture and have a trace of whether the commands had been executed as imagined.

As an educational community developed around Logo, its use moved beyond programming towards the development of constrained, playful environments in which students could explore powerful ideas of mathematics and science. These environments were termed “microworlds,” and like Einstein’s famous \textit{gedanken} (thought) experiments, rendered technical ideas in a form conducive to playful engagement with fundamental ideas. Important principles [29] included: putting learning into children’s hands, that is, treating

\(^1\) We use "SimCalc" to refer to the project and SimCalc Mathworlds® to refer to the implementation.
them as *bricoleurs* (tinkers) and letting them create; seeking newly accessible ways to render powerful ideas in an experience students could interact with; and “no threshold, no ceiling” environments which were initially simple but allowed engagement, over time, with complex endeavors [1].

Whereas the design target in earlier CAI systems was usually a course of study, the design of microworlds often began by identifying a foundational concept of science or mathematics which students were not reliably learning in a traditional course. Design work included the identification of foundational concepts and thinking out exactly how to invite and encourage engagement with those ideas, creating a kind of playground in which the learner would be brought back to them time-and-again. With these playgrounds, modeling was often a fundamental activity: students were invited to use scientific or mathematical constructs to reproduce a familiar phenomena or experience. For example, students might use the ability of a turtle to move forward and turn in small increments to model a circle as the limit of a regular polygon with increasingly short sides and small turns. Elements that are now often brought into discussion of learning strategies were taken as foundational. In particular, embodied learning and the use of virtual---and physical---manipulatives.

Over the years, there have been hundreds of implementations of micro-worlds in different areas of endeavor, ranging from music [2,3] through to chemistry [53] and physics [62]; it continues in projects such as those reported in diSessa’s Changing Minds [10], which focuses on bringing children into contact with powerful ideas, and has found new life in a variety of intellectual homes: via the Scratch language (http://scratch.mit.edu/), in Media Computation (http://coweb.cc.gatech.edu/mediaComp-teach, 17), and Storytelling Alice [37]; through many manifestations of computationally controllable objects (c.f. 28, 61); in more complex programming environments such as the parallel, distributed environment of NetLogo [63]; and in game design (c.f. 47); however, some of the most profound, long-lasting and widespread have been in the area of mathematics education.

I. B. The Context of Mathematical Instruction

In the 1980s, the possibility of new ways of engaging students with mathematical ideas began to intersect with a movement towards reform of mathematical curricula. Just as the launch of Sputnik in the 1960s gave rise to “new math,” the influential report “A nation at risk: The imperative for educational reform” [16] incited a wave of thinking about the future of mathematics education. Whereas in the early years of the 20th century, educators sought to enable all students to master shopkeeper arithmetic, now the focus began to shift to algebra for all – a dramatic increase in instructional challenge. Simultaneously, mathematics educators began to question whether educational goals should be limited to computational and symbol manipulation skill and pushed for mathematical attainment to include conceptual understanding and mathematical practices (such as expressing generalities). Presently, this shift continues with newer curriculum standards emphasize not just skillful and accurate execution of mathematical calculations and procedures, but also focusing on conceptual development and enculturation into mathematical practices (e.g., the Common Core State Standards for Mathematics).
This shift in educational goals was supported by emerging mathematics education research, which was grounded in developmental and cognitive science approaches. Unlike instructional research, which tends to ask: “does this or that teaching strategy produce greater test score gains?”, the newly emerging body of mathematics education research studied how individual learners build the next stage of mathematical thinking upon ideas and competencies they already had. This research was represented, for example, by the scholarly society “Psychology in Mathematics Education” and interlinked with the policy prerogatives noted above through the agency of an association of mathematics teachers, the National Council of Teachers of Mathematics. In some sense, this scholarship starts from the ur-question of why it is so difficult for so many people to learn mathematical concepts that are quite plain to those who already know them. The kinds of answers provided have to do with uncovering the detailed hidden entailments of mathematical thinking and the aspects of human psychology that make representations work or, sometimes, not work for particular learners as particular developmental moments.

**Dynamic Representations**

SimCalc Mathworlds® ([http://www.kaputcenter.umassd.edu/products/software/](http://www.kaputcenter.umassd.edu/products/software/)) constitutes one of a number of technologies for learning that dovetailed with and elaborated the opportunities for reform of mathematics within the context of a pre-existing body of scholarly thought about mathematics education. Other similar approaches which emerged at roughly the same time include Geometer Sketchpad [30] and Cabri Geometre [39]. This class of technology eventually became known for its “dynamic representation” approach. Like Logo, dynamic representations enabled learners to be active, playful, constructive, and expressive in a computer-based medium. But unlike Logo, dynamic representations do not focus on programming. Like microworlds, dynamic representations provide an invented, pedagogical environment that is meant to engage students with fundamental ideas of mathematics, rendered in an interactive and dynamic form. Relevant to the emergent development and cognitive psychology of the time, both microworlds and dynamic representations intend to activate students’ prior knowledge, and through the activities of exploring and constructing, allow students to build new knowledge. However, whereas microworlds have somewhat more focus on a fanciful context for mathematical ideas, dynamic representations have more focus on providing interactive mathematical notions and representations.

**Democratizing Access to Calculus: The Mathematics of Change and Variation**

The overall educational purpose of SimCalc was, in Jim Kaput’s, its progenitor’s, words, to *democratize access to Calculus*. At the time that the SimCalc Mathworlds project started, it was clear that the rate of change, co-variation, accumulation, approximation, continuity, and limits were arguably some of the topics that would be most important to children moving forward. Kaput was fond of arguing that whereas “algebra for all” was a necessary advance in educational goals for society in the 20th century, “calculus for all” would be a necessary advance in the 21st century due to the importance of mathematics in understanding and regulating processes of change. Importantly, Kaput conceptualized Calculus not as a course of study taken at the end of a long sequence of mathematical prerequisites, but rather as a strand of mathematical thinking that could develop begin-
ning as early as elementary school and which could enrich classic middle school topics, such as proportionality. Thus Kaput used the phrase “mathematics of change and variation” (MCV) to break the mindset of Calculus as a specific course, and to instead focus on how the underlying ideas could develop over a decade or more of a students’ mathematical development [35].

At the heart of the SimCalc approach to MCV is the idea of considering rate as the relative change of two quantities (for example, position and time) which could be represented as the slope of a graph or a parameter in an algebraic expression or a motion or a set of values in a table. Technology provided a technical affordance for realizing these representations in a dynamic interactive form. Pedagogical and curricular research sought to exploit this technology to allow a potential restructuring of when mathematical ideas could be explored by young students as well as upper high school. Introducing a dynamic, technological medium also allowed young children easy access to touch and manipulate mathematical objects, including moving pieces of graphs and watching the resulting changes to the movement of one to linked actors in a simulation. Later on in the evolution of the SimCalc program of research and development, the affordances of classroom networks were incorporated into the integrated software/curriculum suite of resources to enable students to make personal mathematical constructions that could be shared within the classroom and publically displayed by the teacher in many different configurations. This allowed some researchers to not only investigate the cognitive dimensions of learning the MCV with diverse populations of students but also affective dimensions of engagement and motivation as the participatory nature of the classroom changed [6].

The aim of this program of design and research, “democratizing access,” diverged from the contemporary emphasis on raising test scores, because Kaput sought to introduce students to concepts which were not commonly measured on tests – and to focus on conceptual understanding, whereas most assessments measure procedural skill. It also diverged from an emphasis on preparing students to use modern workplace tools, such as spreadsheets, by focusing more on mathematical insight than on mathematical applications. “Access” did not mean availability or affordability of technologies or textbooks, but rather access to meaningful opportunities to learn. Operationally, “democratizing” meant an emphasis on design and development of activities for students who would ordinarily be excluded from reaching a traditional Calculus course by deciding “I’m no good at math” or by not achieving suitable grades in prerequisites.

To achieve democratization of access, Kaput was always committed to the idea that technology and curriculum should be, indeed, had to be, co-developed to better build on learner strengths. Additionally, he was always committed to classroom-based education; classrooms are places where all students can have an opportunity to learn (overcoming, for instance, limitations of the resources available in their homes) and where socialization into a mathematical culture can occur. Emphasizing classroom-based education has consequences. First, it means that design must address the situation of having a teacher together with a group of students as well as the situation of the individual learner. Second, it means that (truly) no learner can be left behind.
These braids of thought, stemming from the potential of the computer, the detailed examination of the cognitive bases of mathematical knowledge, a commitment to classroom-based education led to a formulation of the SimCalc research project as *restricturing knowing*, finding points of possible design action where learners’ strengths, representational affordances, and reorganized curriculum provide opportunity. Some of this history has been reported, particularly in [50], which focuses on the research (rather than the design) trajectory of the project and itself draws on and summarizes diverse earlier sources, including [31, 34, 43, 44, 49].

Two of the three authors of this retrospective analysis started their careers working with Logo and Boxer, an important rethinking of Logo that integrated programming, specific microworlds and hypertext to create a multi-purpose computational medium [10-12]. Based on this preparation, they recognized the great opportunity inherent in SimCalc. The first author was involved in several projects, primarily from 2000 until 2008 and is currently a professor of computer science with a focus on the design of systems that restructure knowing. The second author brings expertise both in computation and the learning sciences and has the longest history with SimCalc, running from 1994 until the present time and encompassing every aspect of the project from implementation to scaling. The third brought a background in mathematics and mathematics education to bear on the project starting in 2000 and took on the running of the overarching project after Kaput’s sudden and untimely death in 2005.

II. Two Descriptions of SimCalc Designs

The issue of how the design of SimCalc technologies is described has depended on the context of the description and on the unfolding of projects that have themselves depended on opportunistic factors such as the particulars of novel technologies, shifts in policy concerns, alignments with school districts, teachers and curricula, funding opportunities, development of thought about pedagogical leverage. Indeed, we prefer to think of it as a representational infrastructure or set of design principles that are and could be used in other mathematics software [23, 45]. Under this rubric, a wide range of functions have been investigated (including new curricular materials available at: http://www.kaputcenter.umassd.edu/products/curriculum_new/).

Nonetheless, in a major 2010 paper reporting the use of SimCalc in three large-scale randomized trials, the technology is described as follows, with five components:

1. Anchoring students’ efforts to make sense of conceptually rich mathematics in their experience of familiar motions, which are portrayed as computer animations;
2. Engaging students in activities to make and analyze graphs that control animations;
3. Introducing piecewise linear functions as models of everyday situations with changing rates;
4. Connecting students’ mathematical understanding of rate and proportionality across key mathematical representations (algebraic expressions, tables, graphs) and familiar representations (narrative stories and animations of motion); and
5. Structuring pedagogy around a cycle that asks students to make predictions, compare their predictions with mathematical reality, and explain any differences.

These components are explained as follows:

The SimCalc MathWorlds software provides a “representational infrastructure” (Kaput et al. 2007; Kaput & Roschelle, 1998) that is central to enabling this approach. Most distinctively, the software presents animations of motion (Figure 1). Students can control the motions of animated characters by building and editing mathematical functions in either graphical or algebraic forms. After editing the functions, students can press a play button to see the corresponding animation. Functions can be displayed in algebraic, graphical, and tabular form, and students are often asked to tell stories that correspond to the functions (and animations)…. In addition to proportional and linear functions, students and teachers can make piecewise linear functions, which can be used to model familiar situations.

This description is accompanied by a picture:
Figure 1: A picture of a SimCalc Mathworlds microworld. A position graph is shown related to the simulated situation shown in the “world” portion of the screen. The manipulation and animation functions are set in a window below. Playing the animation causes both the sweeping out of time on the position graph and the animation of the characters in the world. The motion of the character with the orange shirt is described by the orange line while the motion of the character with the purple shirt is described by the purple line.

These five definitional elements were largely present in Kaput’s 1994 description of what would become SimCalc, yet the description of the user experience is quite different from what eventually transpired:

“Imagine a pair of 12-year old students driving a computer-simulated vehicle that provides a windshield view and a carefully linked user- or system-configurable collection of data displays for the dashboard; one set of displays for time, another for velocity, and a third for position. These include sounds for each set (metronome for time, engine pitch for velocity and “echo” when passing roadside objects for position). The dashboard display can include velocity and/or position versus time graphs generated in “real-time” as well as clocks, odometers, tables and so forth. This “MathCars” system is designed to help link the phenomenologically rich everyday experience of motion in a vehicle to more structured and formal representations and to provide exciting and intensely experienced contexts for reasoning about change, accumulation and relations between them.
After some unstructured driving trips, they are now planning to follow a school bus whose (highly variable) velocity has been specified beforehand based on (one-dimensional) velocity data they collected on their own bus trip home the day before….” [32, p. 391]

Aspects of this vision for the design appear in an even earlier picture as reproduced in Figure 2 [33, p. 540]:

![Figure 2: An Early Envisionment of SimCalc Mathworlds](image)

The major focus in this view is on authenticity of the motion phenomenon. However, the design focus in 2010 paper is also mentioned:

“They will also set up and run simulated “ToyCars” on parallel tracks to study relative motion more systematically, describing the motion of each algebraically, confronting such questions as how to describe a later start versus describing a simultaneous start but from different locations….” [33, p.392]

These two descriptions represent both views of the opportunity space, and perspectives on what constitutes research on learning. In fact, Kaput’s 1994 [32] article reads like a mathematical proof. It makes arguments for a set of apparently disconnected beliefs and circumstances, taking particular care to unpack the relationship between the child’s physical interactions with the world, the child’s experience of physical interactions and the mathematician’s formalisms. It then assembles the findings into the vision quoted above. The 2010 description is a starting place for exploration of how a particular implementation of “technology + curriculum” fares as it faces the world.
Some key design differences between SimCalc as Kaput first imagined it and SimCalc Mathworlds as realized through a process of considerable design research include:

- The representation of motion shifted from a 1st person (“point of view”) perspective to a 3rd person, flattened perspective. Although the 1st person view is experientially compelling, it was hard for students to make connections between distance in a graph and distance in a windshield view.
- The students’ opportunity for control shifted from controlling via a gas pedal and brake to control by changing the graph itself (as indicated by the square control points on the graph in Figure 1). This followed the realization that by giving students the ability to construct the more mathematical representation (rather than just see it as an output) they could better come to understand what it meant. The output became the movement of the soccer players.
- The nature of the mathematical function changed from a curve to a piecewise linear function. This reflected understanding that curves were cognitively difficult objects for students to make sense of and that the learning progression could eventually get to curves from piecewise functions by showing how functions made of smaller and smaller pieces could come to approximate curves.
- There is also a noticeable simplification in the number of display elements in the eventual SimCalc Mathworlds design, reflecting the insight that it was essential to focus the learner’s attention on a few representations at a time.

There is one contrast between these images that is not indicative of a design change: one image shows a velocity graph and the other shows a position graph. SimCalc MathWorlds has always had activities with both velocity and position graphs.

Figures 1 and 2 present a snapshot of changes from 1992/1994 until 2010; however, the contrast does not adequately explicate the nature of the design. Hence, we now move to a broader overview of the set of SimCalc projects and the design thinking that emerged in them.

It worth noting here, though, that the design principles have been complemented by a larger implementation principle when SimCalc is introduced into large number of classrooms. The larger implementation principle is to present teachers with an integrated system of the software, curricular workbooks, and teacher professional development. The achievement of a stable learning effect when SimCalc is introduced in hundreds of classrooms is importantly NOT due only to software features. Rather, it also is the consequence of carefully designed workbooks that lead teachers and students through a curricular learning progression with the software (and including exercises and discussions without the software) and is a consequence of teacher professional development that encultures teachers into appropriate classroom use of the software.

III. An Overview of SimCalc Projects
The larger SimCalc project starts from an ideal of improving mathematical teaching and learning, a mechanism, the computer, and a series of perceptions about learners and learners. It developed into a family of projects, each of which explored a facet or aspect of the whole. One history of the effort is given in [50], with a focus on the processing of going from small design studies to larger classroom tests.

**Planning**

An initial planning period roughly from 1992-1994 was concerned with the examination of curriculum, the history of mathematical thought and a review of the learning sciences. Also during this time, several years were spent conducting the microanalysis of very small numbers of students working with different designs for the representations.

**First Iteration**

The first software design was implemented and then abandoned after less than a year. This design had an overarching narrative concept called “Alien Elevators” and was an extended game in which students would infer rules by which elevators were controlled on an alien planet, where the elevator buttons controlled velocity, not the target floor. This was abandoned because it was found in user testing that the story distracted students from the mathematics and the interface did not yield mathematical insights for students. However, one component of the interface was very productive for students and the project moved forward focused on this element. The element that was retained was a representation of velocity on a graph as a step function, where each step specified a constant velocity for a duration of time.

The work of this time consistent of experiments with a small number of students in a lab, or short teaching experiments in a classroom, each examining how the emerging SimCalc Mathworlds dynamic representations could enable students to develop particular target mathematical understandings.

**Working with Teachers and Classrooms**

Some of this kind of work continued, especially that led by Ricardo Nemirovsky [41-44], Helen Doerr [14], Janet Bowers [41, 43], Roberta Schorr [36], and Walter Stroup [63], all of whom continued to examine micro-level learning interactions as well as small scale teaching experiments. Nemirovsky in particular, kept an emphasis on embodied aspects of interaction, continuing to address the problem of how the child’s real physical experience of movement, representations of physical experience, and mathematical representations conjoin.

This was followed by a second phase, three years spent on curriculum, which involved different educational settings and partners---in Newark, NJ, Syracuse, NY and San Diego, California. Noteably, sites were chosen to include diverse students who would not ordinarily go on to study Calculus. In addition, tests were conducted with students at different grades, include middle school, high school, and early undergraduate years. At this time----while many of the ideas were beginning to gel but not yet set----there was enough stability to involve teachers and classrooms full of students. However, even after nine years of work and development, measurement of learning outcomes only used research-
designed pre-test/post-test assessments, consisting of items as created in response to the special purposes of the particular innovations. These projects resulted in an important diversity of curricular materials, variations of the software, and test questions.

**Three Parallel Investigations**

At this point, nine years into the project, it split. There was good reason to believe that the core ideas were solid but it was not clear how they could become widely used. Three avenues were explored: technological, political and scientific.

The technological exploration started from the observation that, although most students had theoretical access to computers, only graphing calculators received widespread, frequent use. Therefore, the research turned to how smaller, less-expensive devices could be used to make the key affordances available. The small size of the devices pushed the research to explore distributed, social, networked activities [20-22, 24, 26, 6]. Some of this work was supported by Texas Instruments, building on a network infrastructure that they were developing and teacher professional development facilities that they supported. Work with the graphing calculator was awkward because the screen was small and low resolution and calculator keys had to be repurposed to implement SimCalc Mathworlds functions. Therefore another avenue was also explored: the then novel (and now defunct!) Personal Data Assistant, in particular, Palm Pilots. These devices provided infrared beaming, a low-overhead technology nicely suited to classroom communication [58-60]. These projects led to the design of distributed activities that were social and fun but that always drew the student’s attention back to important and difficult mathematics.

The political development had to do with influencing the key state mathematics examinations in Massachusetts. In particular, Kaput’s influence over the construction of the high-stakes examinations resulted in a more rational and principled framework. Kaput and Hegedus additionally worked on the construction of SimCalc-based curricula, which they conceived of as a progression throughout middle and high school (http://www.kaputcenter.umassd.edu/products/curriculum_new/).

The third avenue was scientific demonstration. From 2000 until 2008, culminating in the 2010 paper, the project planned and then conducted a series of large-scale experiments, including randomized trials [51, 54, 57]. More than two thousand students, and 150 schools were involved. A pilot plus three different experiments with 7th and 8th graders in Texas demonstrated and replicated that SimCalc Mathworlds could produce significant learning gains in important mathematical concepts across a wide-range of teaching circumstances.

This level of demonstration was a triumph and should be seen not just as a confirmation SimCalc Mathworlds itself but also of the design-based research methods used at different scales throughout the early phases of the project. Such methods are necessarily complex, require intense scholarship and can lead to substantial setbacks, as in the initial implementation; however, they can report real and important learning changes. The success of SimCalc at scale confirms the importance of support for the slow accretion of knowledge about learning and educational change.
The assessments used in these experiments are one example of an element that only worked because of diversity and persistence in the prior work. These experimental work spent over $1,000,000 developing assessments that were both sensitive to the intervention, spoke to teacher and administrator concerns about curriculum, had the right reading and cultural properties, and could be administered within a classroom period. However, none of this development could have happened without pre-existing theories of learning and the roughly 700 test items gleaned from the classroom work over the years (as well as other scholarly studies of algebra learning also primarily supported by the National Science Foundation).

The Changing Landscape

In theory, the technological, political and scientific elements of the project could have been more substantially supplemented by a fourth element, an economic strategy. Indeed, work with Texas Instruments moved in that direction and resulting, in part, in the TI nSpire handheld device. This device does incorporate dynamic representations, particular for geometry, graphing, and data; however, it stopped short of including SimCalc representations such as motion and editable piecewise graphs. Further, other dynamic representation-based projects, such as The Geometer’s Sketchpad, did pursue and succeed as business ventures (for a time); eventually, The Geometer’s Sketchpad was undermined by changes in the market and by the availability of a free, open-source clone. Furthermore, at the very time that these projects were attempting to improve and widen instruction in mathematics, policies such as No Child Left Behind (http://www2.ed.gov/nclb/landing.jhtml) were in essence causing teachers and districts to become more risk-averse [7]. It may well be that economically self-sustaining models of adoption are an unrealistic burden on an intervention aimed at changing so many elements of existing practice at the intimate level of the learning invisible from outside the classroom.

IV. Design Rationale

With this overview of the construction and development of the family of projects until 2008, we consider the rationale behind and the implementation of each element of the 2010 definition. These are the elements that, as the larger project has developed, have become assumptions in research papers. Yet the pedagogic opportunity lies in the details of how these elements are supported by the technology, and understood and utilized in the classroom. These are the elements that each teacher, curriculum designer, assessment creator, and technology designer needs to grapple with.

1. Anchoring students’ efforts to make sense of conceptually rich mathematics in their experience of familiar motions, which are portrayed as computer animations;

Kaput approached the ideas that would become SimCalc though historical, curricular and literature analysis. By the end of the 1980’s, some kinds of computational environments for learning allowed learners to connect formal algebraic expressions with graphical rep-
resentations, so that the learner could “follow along.” Building on prior work examining mathematical representations, and starting as early as a 1992 publication [33], Kaput identified the context of motion as missing from instruction in Calculus (and algebra).

Although the development of calculus was historically motivated by the desire to describe motion phenomena, instruction had virtually no relationship to authentic context that motivated the work itself. Indeed, often teachers explicitly reject the idea of using motion to introduce algebraic or calculus on the grounds that motion is not mathematics. It is, instead, physics. But motion is not only an academic topic that must be described in formal terms. It is also a universal human experience.

What Kaput perceived was that integrating familiar aspects of motion into mathematics instruction could benefit students by allowing the redistribution of “sources of structure and action from the mental to the physical realm.” [32, p. 394]

However, Kaput’s initial thought about how to implement this experience evolved through small group work with children. The elements of realism that were featured so vividly in the original description were refined into more strategic and abstracted representation of motion seen in the “world” graphics. Although the earliest work resembled current approaches to games and game-like environments, this was soon dropped. While capturing student interest and engagement is important, and using existing student strengths, such as their experience of the natural world, is crucial, the experiences must not overwhelm or downplay the mathematics to be learned.

The decision to implement motion as animations or depictions in an artificial “world” allowed the scope of inquiry to be simplified to the representation of movement along a line (or, better yet, a number line!). This simplification created a congruence between the portion of motion depicted in the system and that actually modeled in high school algebra.

Other aspects of the early vision did not make it into the branch of exploration expressed in the 2010 description, but by-and-large for pragmatic rather than pedagogical reasons.

2. Engaging students in activities to make and analyze graphs that control animations;

Using proper notation is a metric of understanding of algebra and calculus; therefore, use of that notation is usually prioritized in instruction. Yet, arguably, the roots of student

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2 For example, a branch of pedagogical exploration considered collecting real data using motion detectors (ultrasound sensors from cameras), but did not become part of the main-stream project, because it would have introduce another object for schools to purchase. Yet, the exploration of physically-embodied phenomena and varieties of mathematical notations, and the use of hybrid physical/cybernetic devices embodying dynamical systems continued and continue (Brady, C. (2013). Perspectives in Motion (Unpublished doctoral dissertation). University of Massachusetts, Dartmouth, MA).
understanding lie not in algebra, but in the depiction of what is important about the motion phenomena. Graphical representations are less compact than algebraic ones; however, graphs are a more common, everyday representation. For example, one often sees graphs in the newspaper, but hardly ever sees algebra in the newspaper.

In particular, like the motion itself, graphs can be animated over time. Contrast between the depiction of the motion in the “world” and the depiction of the graph help students learn how a graph represents. The graph is an abstraction, that, by its nature draws our attention to certain aspects of motion (change in position over time, change in velocity over time) which we suggest are important.

The context of motion implicitly suggests that time might be an important thing to think about. The everyday description of motion utilizes time. The graphical representation of motion makes time an explicit element. But when we make time explicit element, we change what we are showing about position, compared to an actual depiction of that motion. Thus, a graph, such as shown in Figure 1, does not show a picture of a person going up and down hills but rather a metric of the person’s displacement in two-dimensions.

This point often seems simple to those who already know algebra, but it is hard to overestimate its importance to large categories of learners. In recent years, research has suggested that people differ in their ability to interpret different kinds of information. Some people, those who prefer linguistic kind of information, do well with current teaching and learning practices. Others already bring to the project of learning mathematics, a tendency to interpret depictions as symbolic. But some subset of students tend to see graphs as pictures [27, 38]. Even those who are inclined to see graphs as symbolic representations may become confused about the nature of a particular representation.

Engagement with how a graph represents was enabled by a core innovation in the SimCalc approach: to make and edit graphs without having to edit the algebraic notation, and this gave students an easy way into manipulating mathematical representations of motion. Indeed, in the SimCalc approach, students often learn about graphs and tables before they encounter algebraic notation. Rather, than starting with formal symbolic notations, students mathematical experiences are gradually formalized.

Animating graphs interacted with design decisions in ways that were not central to the mathematics but that were central to the HCI and pedagogical usability of the system. Drag-and-drop facilities meant that many graphs could be made and their motion consequences easily explored. One key interface element in enabling easy exploration was the implementation of snap-to-grid “hot spots” that allowed students to easily explore integer end-points. This was controversial because it compromised continuity, which is an important mathematical concept. However, pragmatically, trying to make lines do exactly the right thing can be a time consuming distraction. The downloadable version of SimCalc Mathworlds (http://www.kaputcenter.umassd.edu/products/software/) allows users to turn off snap-to-grid facilities. Another set of difficult usability issues had to do with the relationship between grabbing and pulling function lines as compared to changing axes or labels in the world’s ruler or the Cartesian coordinate plane.
These two elements, animation and tying animation to easily manipulated graphical representations, can lead to subtle but important curricular changes. In particular, one place that we lose active cognition amongst students is in introducing the idea of slope. Most students learn the slope of a line as a calculation of “rise over run”, often fixating on the identification of points that make the actual calculation easy. The slope then becomes one calculation among many, a calculation that, for mysterious reasons, is sometimes negative. SimCalc allows the teacher to ground an understanding of slope in a far more sophisticated context, a context in which rate is demonstrated to be instantaneous as it sweeps out, connecting the characters’ motions with their positions at a given time. Conceptualizing slope as a description of the relationship between time and position leads towards calculus without demanding the mastery of algebra and grounds the concept of negative slope as “going backwards”.

Another kind of curricular change permitted by SimCalc representations is the more coherent presentation of proportion as a reduced case of rate in which the line just happens to go through 0. Proportion is a major middle school topic, but often is presented merely as a “calculate the missing quantity” problem, where three numbers are given and the fourth must be calculated using the formula “a/b = c/d.” Of course, this formula can be useful to permit calculations to figure out how much 5 pounds of potatoes will cost if potatoes are $3/2 pounds. However, it is also a mathematical dead end – it doesn’t lead anywhere in further mathematics. SimCalc represents proportion instead as a constant of proportionality, k, in y = kx, which is the slope of a line. The analysis of slope as a ratio, k = y/k, and a proportional function as a simple case of a linear function, allows a trajectory of mathematical development that continues from middle school through calculus.

3. Introducing piecewise linear functions as models of everyday situations with changing rates;

Traditional instruction in algebra and calculus emphasized the definition of a function and the importance of continuity in the definition of a function. The continuity assumption is key to the ability to calculate inherent in Calculus. But those students who went on to become engineers would go on to use piecewise functions extensively, because many physical systems are best modeled not as one continuous curve but as discontinuous segments that may each be represented as linear (at least well enough for their engineering purposes).

All motion of an object over time is continuous. However, people’s experience of motion is not continuous. One of Kaput’s major insights from the first iteration was that, by introducing piecewise linear graphs earlier and delaying the introduction of the idea of continuity, many important ideas could be introduced earlier and more effectively into the curriculum to a wider range of students. This notion insight utilizes the principle of building on existing student strengths, but it is legitimized by engineering practices.

This representation was easy for students to control, by adjusting the height and width of rectangle “chunks” of velocity (where the height was speed, the width was time, and the
area represented change-in-position). It also turned out that students could easily understand the area as position change, and this led to interesting mathematical challenges, such as finding different ways to move 6 meters (see Figure 3).

Figure 3: Changes in velocity: different ways to move six meters.

Further, the velocity graphs could be related to piecewise position graphs, which were also found to be productive in terms of student insight. Figure 4, for example, represents a complex way to get to a final position of 6 meters, but with changing speed and backwards motion.

Figure 4: Piecewise graphs are easier for children to understand than continuous ones.

The introduction and prioritization of graphical experience with piece-wise linear models is the cross-cutting computational, pedagogic, and conceptual insight that democratizes access to the math of change and variation. By enabling students to work with piecewise linear functions, that is, functions over a limited domain, we can let them explore the descriptive properties of the mathematical language we are introducing first, before showing them that the physical experiences of motion that arise as a consequence of being
human are not precisely what Newton was modeling in creating that mathematical language.

A key activity used in conjunction with piecewise linear functions is the “exciting sack race” lesson. This is the more developed form of “driving behind a school bus” from the original conception. The students are given or create one function representing a person who runs a race at a constant speed over some domain (the straight line in Figure 1). They then have to create another line, representing a “crazy” race---like a sack race---putting together functions piecewise on the graph. The only rules are the race must start at time=0 and end in a tie. Subsequently, students are asked to write (in words) the story of the race. Often these stories are on the order of “Jane started really fast, but then she realized that she had forgotten her sunglasses so she ran back to the starting line to get them, but by then she was so exhausted that she couldn’t run as fast, so she moved along, but she started staggering and being confused and sometimes went backwards until she finally stopped for a while. That gave her a rest, so she finished really quickly.” Some or all students read their stories aloud, while the class looks at their graphs. Sometimes, teachers ask students to exchange these stories, and as the new person to draw the function line from just the story. Then the students compare the original and the new lines. This is usually fun, but also motivates the future use of more precise, specialized mathematical language.

In particular, by starting with graphs as compared to algebraic expressions, by tying those graphs to motion phenomenon and finally by allowing the graphs and the motion to model complex motion phenomenon, student learning can be grounded in the desire to represent tractable and interesting problems. This enables complex material to be taught significantly earlier in the curriculum, and indeed elements of the core idea in SimCalc have been taught as early as 5th grade (9-10 years old).

Both the inclusion of motion phenomenon in mathematics and the emphasis on animating graphs allows changes to the curriculum and changes in emphasis within existing curricula. However, the introduction of piecewise linear functions requires a shift in thought about the core material taught in Calculus. The graphical, piecewise approach motivates the more succinct expressions found in Algebra and Calculus.

4. Connecting students’ mathematical understanding of rate and proportionality across key mathematical representations (algebraic expressions, tables, graphs) and familiar representations (narrative stories and animations of motion);

SimCalc turns algebra upside down by introducing piecewise functions early and also by introducing graphical interpretations of rate and proportionality, and allowing grounded explorations of slope. Graphical understandings are important in their own right, but they are also important in two other ways: as a pathway to other (algebraic) understandings and as a pathway towards understanding the system that compromises the mathematics of change and variation. Each kind of representation---computer models of the “world”, graphs, tables and algebraic expressions, even word-based stories---emphasizes different
aspects of the system, some of which are more usefully mathematical. Exercises that ask students to move across different representations develop fluidity and familiarity.

One could say that SimCalc projects a different image of what it means to “know” algebra. In a traditional symbolic approach, knowing algebra is often tantamount to knowing the grammatical transformation rules that correctly re-write one expression into another form. SimCalc still honors this as important, but aligns with an image of “knowing” which has to do with connections among representations. In this view, “knowing” a concept like rate means being able to coherent trace the connections of the concept in different forms – to be able to see rate an experienced speed, a slope of a graph, covariation in a table of number pairs, and in a symbolic form. For students to build this connected sense of “knowing” algebra, they need tools which help them make the connections. SimCalc is squarely aimed at this connected epistemology.

One of the design challenges associated with this epistemology is that students cannot reasonably make all connections, all at once. Thus connections among representations must be introduced gradually, which corresponds to giving students access to different visual representations only as the master prior representations. In the early versions of SimCalc Mathworlds, this was handled by developing generic and powerful software which could show all possible representational forms, but configuring the software in saved documents. Teachers and students could then load documents in a sequence corresponding to the learning progression in a curricular workbook. More recently, the application/documents approach has been superceded with cloud-based solutions which deliver variant representations to students through activities arranged in a playlist.

5. Structuring pedagogy around a cycle that asks students to make predictions, compare their predictions with mathematical reality, and explain any differences.

This concept was always inherent in SimCalc Mathworlds in that the point is draw student attention to aspects of the world and models that they might otherwise overlook. However, the idea of an explicit cycle of comparative prediction centered on the problem at hand developed slowly over time, and in conjunction with other related theorizing. One highly related pedagogical move is articulated by Schwartz and Bransford [5, 55] as contrasting cases. The chief idea is that the designer or teacher creates a situation that makes the problem that will eventually be solved in the lesson clear before offering the solution.

The original conception of SimCalc was that the children would explore. Enabling exploration is still a key principle. However, exploration by itself does not necessarily lead to learning – for example, students may get to “solution states” for a particular challenge by exploration, but may not know how they got there. The predict-compare-explain cycle is meant to engage students in overt planning and reflection, with an eye towards developing stable explanations of the mathematical representations they are using. Further, the cycle plays into conventional classroom structures, where teachers lead discourse and ask
students to make predictions and give explanations, as a way to check for and cultivate desired understandings. The commitment to classroom-based instruction means that the technology and related curriculum must respond to the teacher’s need to ensure that certain material is encountered.

The predict-compare-explain cycle may be used in whole class activities. Additionally, often SimCalc is used with worksheets that ask students, singly or in small groups, to engage in specific activities and record the history of their interaction with the system and the lesson. The cycle is a generally beneficial practice that particularly gives the student helps ensure that the student thinks about and processes what s/he is experiencing.

The development and exploration of this principle is a design response of the SimCalc project to the problem of enabling both structured progress and exploration. It is a design response implemented in curriculum and use-practices rather than in the technology itself. It is thus aligned with some research on classroom orchestration [8,9], but differs from responses that implement process in the technology itself, via scripts and successive disclosure of information [13, 15].

V. Future Exploration

The five elements of the 2010 description and the associated practices are key and enduring pedagogical contributions of the SimCalc project. However, a number of issues have fallen by the wayside, not through any lack of merit.

1. **Physically embodied algebra learning.** SimCalc long anticipated the importance of embodiment in learning, but with the arrival of the DYI movement including Raspberry PI’s, Arduino’s and really inexpensive sensors and actuators, new opportunities are cropping that have a chance of having impact in real K-12 classrooms.

2. **Encompassing curriculum.** When Jim Kaput passed away suddenly in 2005, he and Stephen Hegedus were in the process of creating an ambitious curriculum that reconceptualized the mathematics of change and variation from sixth to twelfth grade. This project was pursued by Hegedus and has to some extent been picked up and continued in the context projects housed at Roschelle’s Center for Technology and Learning, but suffers from the lack of Kaput’s single-minded focus.

3. **Restriction of curricular scope.** In order to encourage teachers and districts to use SimCalc, exploration has focused primarily on a high-impact setting, that of algebra learning. Algebra is indeed very important. It is the gateway course to a four-year college in the United States. Enabling children to learn algebra and learn it well is a therefore a social justice issue that transcends other aspects of education. But Kaput’s vision started with the phrase “democratizing access to the mathematics of change and variation.” The project of creating a development sequence
focused on this mathematics that would grow from middle school through college is as of yet, not complete.

VI. Conclusion: Dynamic Representations and the Problem of Wicked Problems

The fundamental advance in the SimCalc line of work has been to develop a principled design of a dynamic representation system for learning an important and difficult area of mathematics, conduct research deeply interconnecting that design with cognitive, developmental and pedagogical knowledge bases, and further expand the work to educational evaluations that show the learning gains that are achievable in diverse populations at the scale of hundreds of schools and thousands of students. Further, throughout the course of doing this work, the team has been reflective about refining their account of the key principles in the design.

The design story is thus a story of progress, but also a story of how complex the realization of the deep, transformative potential of technology in mathematics learning is. Technology is not a singular, causal factor in promoting learning and design of successful learning experiences but involves interweaving multiple concerns and levels of design.

In his one and only paper, Berkeley architecture professor Horst Rittel [48] advanced the idea of the wicked problem in design. Wicked problems exist in contrast to tame problems. Tame problems (1) have single-valenced solutions and (2) require only that a person figure the solution out. Wicked problems do not necessarily have solutions. Furthermore, wicked problems are such that the exact formulation of the problem is tied to the kinds and ranges of solutions we consider.

One lesson from the SimCalc project is that important problems in education are wicked problems [56]. They must simultaneously determine the utility of technology whose own properties are constitute wicked problems in relationship to curriculum which may designed in many different ways for the purpose of promoting learning---which itself remains ultimately mysterious---in the tremendously complex environment of classrooms and schools. To begin to do this, and to keep the difficult exploration going, requires not just devotion and wide-ranging expertise, but the garnering of funding from sources with different requirements, expertise in the management of teams, and the ability to focus on the whole and the parts at the same time.

In the end, diffusion of innovation is not simple either. Deeply accepting Kaput’s premise of representation change means sometimes NOT addressing today’s curricular expectations and end-of-the-year examinations directly – but can result in changing the sequence of learning, so that learning accomplishments occur in different years and time frames than what is conventionally expected, and in changing the outcomes, which now may not be fully measured by existing examinations. Kaput aimed his designs at addressing long-term societal change, which can mean that the designs do not tackle short-term desires to increase today’s test scores fully – and can result in slow adoption.
And yet, historically we are clearly in the midst of transformation in what people need to know and be able to do to fully participate in an information age economy. Designs which democratize access to ways of thinking and reasoning which have long-term societal value, such as the ability to reason mathematically about change, have a likelihood of long-term societal impact. The opportunity to design effective dynamic representations – representations which express mathematical meaning through interactive, linked, time-based properties and give a wider range of people the opportunity learn and master corresponding ways of reasoning mathematically – is a wicked design problem worth solving.

VII. References


